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THE BENDING PROBLEM OF AXIALLY CONSTRAINED BEAMS ON NONLINEAR ELASTIC FOUNDATIONS

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Abstract—The problem of bending of axially constrained beams on nonlinear Winkler-type elastic foundations is considered. The solution is obtained by an iteration method using Green's functions. Numerical results are presented for uniformly loaded beams hinged at both ends. Axial forces and deflections are given in terms of three dimensionless parameters which depend on the load intensity, foundation characteristics and properties of the beam.

NOMENCLATURE

A Cross-sectional area.
EI Flexural rigidity.
k, n Foundation parameters.
$2c$ Length of the beam.
N Axial force.
r Radius of gyration.
q Load per unit length.
p Reaction intensity in foundation per unit length.
α_0 $\frac{c}{r}$, parameter.
β $\frac{qc^3\alpha_0}{4EI}$, parameter.
ρ $\frac{Nc^2}{4EI}$, parameter.
λ $\sqrt[4]{\left(\frac{kc^4}{4EI}\right)}$, parameter.
γ $\frac{pc^3\alpha_0}{4EI}$, parameter.
n' $\frac{nc}{\alpha_0}$, parameter.
x, y Coordinate system.
η y/c.
ξ x/c.

1. INTRODUCTION

In the analysis of the bending of beams, it is usually assumed that free relative longitudinal motion of the ends of the beam is permitted. In many beam applications such an assumption is not realistic and solutions based on it may, in some cases, lead to grossly inaccurate results. When an axially constrained beam (a beam whose supports are immoveable in the axial direction) is subjected to lateral loads, it undergoes stretching. The axial constraint force developed is not known a priori, since it depends on the deflection curve of the beam. The introduction of such constraint forces in the analysis makes the problem nonlinear; as a result the differential equation for axially constrained beams can best be solved numerically.

Several investigators [1–4] in the past have considered this nonlinear problem for the case of ordinary beams. Recently, analyses have been presented [5, 6] for the case in which the beams are resting on elastic foundations. The elastic foundation considered in [5] and [6] are of the Winkler-type for which the foundation reaction is linearly proportional to the deflection. In general, however, a foundation does not exhibit a linear pressure-displacement relation. Because of the densification of the base, it may be expected that as the displacement continues to increase the foundation reaction will tend to build up more and more rapidly. In the present paper, the effect of such nonlinear behavior of an elastic foundation is considered in the analysis of axially constrained beams. The beams considered are subjected to uniformly distributed loads and are hinged at both ends.

2. ANALYSIS

2.1 *Differential equation*

The beam configuration is as shown in Fig. 1. In the analysis it will be assumed that the axial constraint force N is constant throughout the length of the beam.

The differential equation of bending of beams on elastic foundations is[7],

$$EI\frac{d^4y}{dx^4} - N\frac{d^2y}{dx^2} + p = q, \tag{1}$$

which can be expressed in nondimensional form as,

$$\frac{1}{4}\frac{d^4\eta}{d\xi^4} - \rho\frac{d^2\eta}{d\xi^2} + \frac{\gamma}{\alpha_0} = \frac{\beta}{\alpha_0}. \tag{2}$$

For a linear Winkler-type foundation, reaction intensity per unit length in the foundation is given by,

$$p = ky. \tag{3}$$

This equation can be written in nondimensional form as,

$$\gamma = \lambda^4\eta\alpha_0. \tag{4}$$

A more realistic pressure-displacement relation in conjunction with the Winkler model will be[8],

$$p = \left(\frac{k}{n}\right)\sinh(ny), \tag{5}$$

which can be nondimensionalized as follows:

$$\gamma = \frac{\lambda^4}{n'}\sinh(n'\eta\alpha_0). \tag{6}$$

In equation (5), k and n are parameters to be determined from experimental data on the foundation. It may be noted that in the limit when n tends to zero, the equation (5) reduces to equation (3). The parameter k in such cases becomes same as the modulus of foundation in the linear theory. Once k and n are determined experimentally, then from the properties of the beam, λ and n' can be calculated.

For the nonlinear foundation, equation (2) can be written as

$$\frac{1}{4}\frac{d^4\eta}{d\xi^4} - \rho\frac{d^2\eta}{d\xi^2} + \frac{\lambda^4}{n'\alpha_0}\sinh(n'\eta\alpha_0) = \frac{\beta}{\alpha_0} \tag{7}$$

Since the supports are immoveable, the change in half length of the neutral surface due to bending is approximately given by[1],

$$\Delta c = \frac{1}{2}\int_0^c \left(\frac{dy}{dx}\right)^2 dx. \tag{8}$$

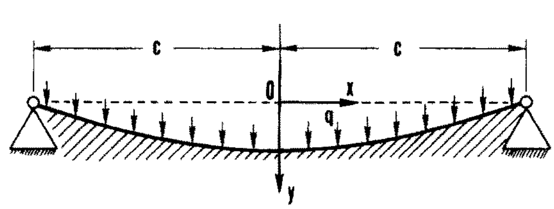

Fig. 1. Axially constrained beam on elastic foundation.

Therefore, the axial force N is given by,

$$N = \frac{AE}{2c} \int_0^c \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 \mathrm{d}x, \tag{9}$$

which can be written in nondimensional form as:

$$\rho = \frac{\alpha_0^2}{8} \int_0^1 \left(\frac{\mathrm{d}\eta}{\mathrm{d}\xi}\right)^2 \mathrm{d}\xi. \tag{10}$$

Solutions of equations (7) and (10), subjected to appropriate boundary conditions will be the solution for uniformly loaded axially constrained beams on nonlinear elastic foundations. For hinged end beams these boundary conditions are,

$$\eta = 0 \quad \text{at} \quad \xi = \pm 1, \tag{11}$$

$$\frac{\mathrm{d}^2\eta}{\mathrm{d}\xi^2} = 0 \quad \text{at} \quad \xi = \pm 1. \tag{12}$$

When axial constraint effect is neglected, the differential equation can be obtained from equation (7) by setting $\rho = 0$,

$$\frac{1}{4}\frac{\mathrm{d}^4\eta}{\mathrm{d}\xi^4} + \frac{\lambda^4}{n'\alpha_0} \sinh(n'\eta\alpha_0) = \frac{\beta}{\alpha_0}. \tag{13}$$

For a linear foundation this equation reduces to,

$$\frac{1}{4}\frac{\mathrm{d}^4\eta}{\mathrm{d}\xi^4} + \lambda^4\eta = \frac{\beta}{\alpha_0}. \tag{14}$$

2.2 *Nonlinear elastic foundation-axial constraint neglected*

The nonlinear differential equation (13) describes the behavior of the beam resting on a nonlinear elastic foundation when the axial constraint effect is neglected. The appropriate boundary conditions on η are given by equations (11) and (12). The solution of equation (13) is given by

$$\alpha_0\eta(\xi) = \int_{-1}^{\xi} G_2(\xi, \psi) f(\psi)\, \mathrm{d}\psi + \int_{\xi}^{1} G_1(\xi, \psi) f(\psi)\, \mathrm{d}\psi \tag{15}$$

where

$$f(\psi) = \frac{4\lambda^4}{n'} \sinh(n'\alpha_0\eta(\psi)) - 4\beta \tag{16}$$

and $G(\xi, \psi)$ is the Green's function in the appropriate intervals [9, 10]. The Green's function in expression (15) is defined such that it satisfies the following differential equation

$$\frac{\mathrm{d}^4 G(\xi, \psi)}{\mathrm{d}\xi^4} + \delta(\xi - \psi) = 0, \tag{17}$$

where δ is the Dirac-delta function. The Green's function satisfies the same boundary conditions as η at $\xi = \pm 1$. It can be shown that such a Green's function is given by

$$G_1(\xi, \psi) = \left(-\frac{1}{6} - \frac{\psi^3}{12} + \frac{\psi^2}{4}\right) + \xi\left(\frac{\psi^2}{4} - \frac{\psi}{6} - \frac{\psi^3}{12}\right) + \xi^2\left(\frac{1}{4} - \frac{\psi}{4}\right) + \xi^3\left(\frac{1}{12} - \frac{\psi}{12}\right), \tag{18a}$$

$-1 \leq \xi \leq \psi$

and

$$G_2(\xi,\psi)=\left(-\frac{1}{6}+\frac{\psi^3}{12}+\frac{\psi^2}{4}\right)+\xi\left(-\frac{\psi^3}{12}-\frac{\psi^2}{4}-\frac{\psi}{6}\right)+\xi^2\left(\frac{1}{4}+\frac{\psi}{4}\right)+\xi^3\left(-\frac{1}{12}-\frac{\psi}{12}\right). \tag{18b}$$

$\psi\le\xi\le 1$

It can easily be verified by substitution that equation (15) indeed satisfies equation (13) and all the boundary conditions on η. Equation (15) is a nonlinear integral equation and has been solved by iteration as discussed in Section 3.

2.3 *Nonlinear elastic foundation-axial constraint included*

The nonlinear differential equation (7) describes the behavior of the beam resting on a nonlinear elastic foundation when axial constraint effects are included. The value of ρ in equation (7) is given by equation (10) and the boundary conditions on η for a simply supported beam are given by equations (11) and (12). The solution of equation (7) is given by

$$\alpha_0\eta(\xi)=\int_{-1}^{\xi}K_2(\xi,\psi)f(\psi)\,\mathrm{d}\psi+\int_{\xi}^{1}K_1(\xi,\psi)f(\psi)\,\mathrm{d}\psi \tag{19}$$

where $K(\xi,\psi)$ is the Green's function in the appropriate intervals and $f(\psi)$ is described by equation (16). The Green's function K is defined such that it satisfies the following differential equation:

$$\frac{\mathrm{d}^4K(\xi,\psi)}{\mathrm{d}\xi^4}-4\rho\frac{\mathrm{d}^2K(\xi,\psi)}{\mathrm{d}\xi^2}+\delta(\xi-\psi)=0 \tag{20}$$

where as before δ is the Dirac-delta function. The boundary conditions on K are similar to those of η at $\xi=\pm1$. It can be shown that such a Green's function is given by

$$K_1(\xi,\psi)=\frac{(\psi-\xi+\xi\psi-1)}{8\rho}+\frac{\sinh 2\sqrt{\rho}(1-\psi)\sinh 2\sqrt{\rho}(1+\xi)}{8\rho^{3/2}\sinh 4\sqrt{\rho}} \tag{21a}$$

$-1\le\xi\le\psi$

and

$$K_2(\xi,\psi)=\frac{(\xi\psi+\xi-\psi-1)}{8\rho}+\frac{\sinh 2\sqrt{\rho}(1+\psi)\sinh 2\sqrt{\rho}(1-\xi)}{8\rho^{3/2}\sinh 4\sqrt{\rho}} \tag{21b}$$

$\psi\le\xi\le 1$

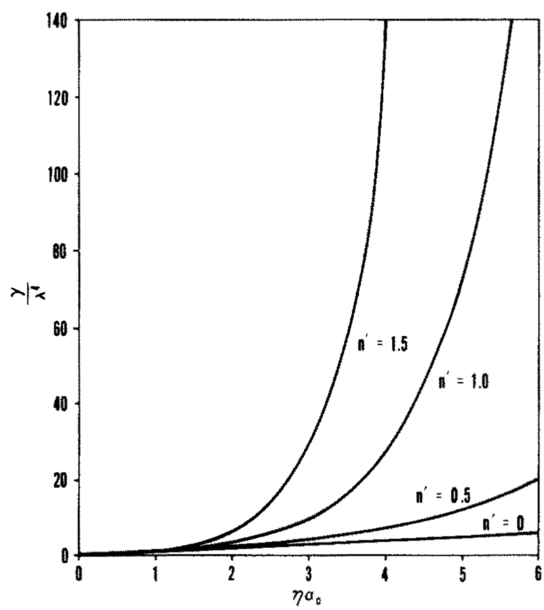

Fig. 2. Pressure-displacement relation of the foundation for various values of the foundation parameter n'.

It can easily be verified by substitution that equation (19) indeed satisfies equation (7) and all the boundary conditions on η. Equation (19) is a nonlinear integral equation and has been solved by iteration as discussed in Section 3.

3. NUMERICAL EXAMPLES AND DISCUSSION

The results presented in Figs. 3–5 for beams on nonlinear foundations are obtained by an iteration method. The interval of ξ is first divided into m parts. In the present paper m is 60 and parts are all equal. The loading parameter β is increased from 0 to a final value (30 in the present case) by increments of $\Delta_i\beta$. These increments need not necessarily be equal. For the initial load $\Delta_1\beta$, the starting values of $\eta\alpha_0$ at the end points of each of m intervals are taken from the known solution of the linear foundation model (without constraint effect). The integrals in equation (15), which represent the solution for beams on nonlinear foundations without axial constraint, are then evaluated by the trapizoidal rule[11] to obtain a new set of $\eta\alpha_0$ values. The new set of $\eta\alpha_0$ values can now be used again to evaluate the integrals and the process will be repeated until the desired convergence is achieved. Next the loading is increased to $\Delta_1\beta + \Delta_2\beta$ and the final values of $\eta\alpha_0$ for the previous loading $\Delta_1\beta$ can now be used as starting values for this iteration and the

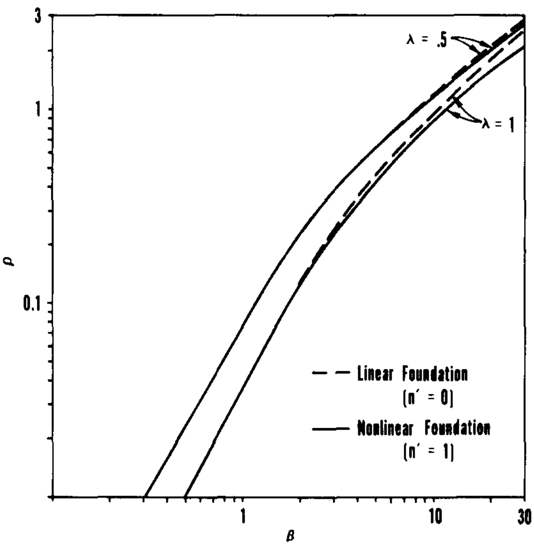

Fig. 3. Axial constraint force vs loading parameter.

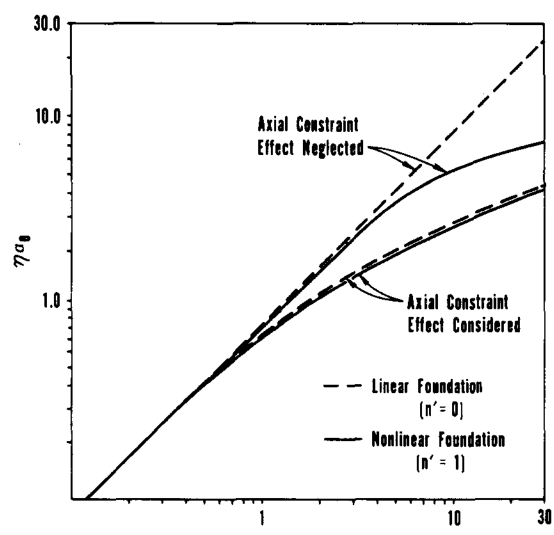

Fig. 4. Midspan displacement vs loading parameter (for hinged end beam and $\lambda = 5$).

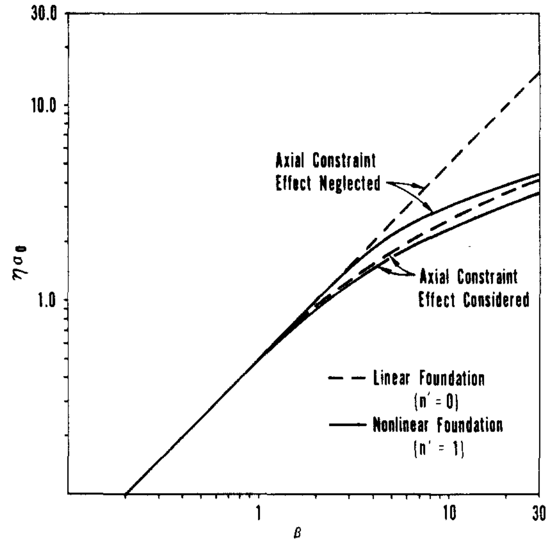

Fig. 5. Midspan displacement vs loading parameter (for hinged end beam and $\lambda = 1$).

process can be continued. It may be noted that computations can be reduced considerably by symmetry considerations.

For the case where the axial constraint effect is included in the analysis, additional computations are required. First, derivatives of $\eta\alpha_0$ have to be determined at the end points of m intervals. This is done by representing $\eta\alpha_0$ by Newton's backward interpolation formula and then differentiating the formula[11]. For numerical computations differences up to the 3rd order are considered. The value of ρ can now be obtained from equation (10) by evaluating the integral by the trapizoidal rule. Integrals in equation (19) can now be evaluated to obtain new set of $\eta\alpha_0$ values and the process can be repeated. Although in the present computational scheme numerical differentiation is used to calculate $d\eta/d\xi$, it is possible to avoid it by differentiating equation (19) to yield,

$$\alpha_0 \frac{d\eta(\xi)}{d\xi} = \int_{-1}^{1} \frac{dK}{d\xi} f(\psi)\, d\psi \tag{22}$$

since K is continuously differentiable. The $d\eta/d\xi$ value from above equation can be utilized to calculate ρ with appropriate modifications in the iteration procedure.

In the present paper iterations are continued until $\eta\alpha_0$ converges within 0·5 percent. It may be mentioned that the success of the iteration depends on the choice of the initial set of values for $\eta\alpha_0$. In the scheme followed in this paper, the initial set of values are obtained from the final values of prior loadings. It is observed that for the cases considered, an increment of $\Delta\beta = 5$ can be made without encountering difficulty in the iteration process. It is also observed that the process of convergence can be improved manyfold if the values of $\eta\alpha_0$ used in the evaluation of integrals are taken as the average of the last two iterated values instead of only the last value. This has been done in this paper and it damps out the oscillations in numerical computations.

Fig. 3 shows the effect of the loading parameter β on the axial constraint force. When $n' \to 0$, i.e. a linear foundation, the results obtained from this analysis match with those of reference (6). It is interesting to note that the difference in ρ vs β curve for the linear foundation model ($n' = 0$) and the nonlinear foundation model with $n' = 1$ is quite small. Figs. 4 and 5 show variations of midspan displacement $\eta\alpha_0$ with β for two different values λ. As before when $n' \to 0$, the results match those of [6].

It may be seen from these figures that although the solution for $\eta\alpha_0$ which ignores the axial constraint effect differs significantly for the linear ($n' = 0$) and the nonlinear model ($n' = 1$) for high values of β, the difference is rather small when axial constraint effect is included. This is understandable, because inclusion of the axial constraint force in the analysis yields significantly lower displacement values as compared to those obtained from the analysis which excludes this force. Since the displacement is smaller, the foundation effect will also be smaller. Therefore, for the purpose of design of axially constrained beams hinged at both ends, when foundation nonlinearity is not very severe (say $n' \leq 1$), it may be assumed to be linear ($n' = 0$) without introducing significant errors.

REFERENCES

1. S. Timoshenko, *Strength of Materials.* 3rd Ed., Vol. 1 and II, Van Nostrand, New York (1957).
2. R. Liebold, Die Durchbiegung einer beidseitig fest eigenspannten Blattfeder. *Z. angew Math. Mech.* **28**, 247 (1948).
3. R. Frisch-Fay, Bar fixed at both ends. *Aust. J. app. Sci.* **12**, 453 (1961).
4. R. Parnes, Bending of an axially constrained beam. *Int. J. mech. Sci.* **13**, 285 (1971).
5. A. K. Niyogi, Bending of axially constrained beams on elastic foundation *Int. J. Mech. Sci.* **15**, 781 (1973).
6. S. DasGupta, Axially constrained beams on elastic foundation. *Int. J. Mech. Sci.* **16**, 305 (1974).
7. M. Hetényi, *Beams on elastic foundation.* The University of Michigan Press, Ann Arbor (1946).
8. A. D. Kerr, Buckling of continuously supported beams. *J. Eng. Mech. Div. Am. Soc. civ. Engrs* **95**, 247 (1969).
9. E. L. Ince, *Ordinary differential equations* pp. 245–278. Dover Publications (1956).
10. F. B. Hildebrand, *Methods of Applied Mathematics.* 2nd Ed., pp. 221–241. Prentice-Hall (1965).
11. J. B. Scarborough, *Numerical Mathematical Analysis.* 6th Ed. The John Hopkins Press, Baltimore (1966).